LINEAR EQUATIONS WITH TWO PARAMETERS*

BY

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The problem of the existence of solutions of two linear self-adjoint differential equations of the second order with two parameters has been treated by Hilbert, † Richardson,‡ and Yoshikawa.§ There are corresponding problems in the theory of linear equations in infinitely many unknowns, and in the theory of linear integral equations. Part I of this paper deals with the first problem, the existence of characteristic numbers λ and μ for the system of equations

$$(1)\qquad \begin{aligned} u_i &= \lambda \sum_{j=1}^{\infty} k_{ij}\, u_j + \mu \sum_{j=1}^{\infty} l_{ij}\, u_j \qquad (i=1,2,\ldots), \\ v_k &= \lambda \sum_{l=1}^{\infty} m_{kl}\, v_l - \mu \sum_{l=1}^{\infty} n_{kl}\, v_l \qquad (k=1,2,\ldots), \end{aligned}$$

where the matrices are subject to certain conditions (§1), and the expansion of the determinant matrix in terms of the solutions. Part II deals with the second problem, the existence of characteristic numbers λ and μ for the equations

$$u(x) = \lambda \int_a^b K(x,y)\, u(y)\, dy + \mu \int_a^b L(x,y)\, u(y)\, dy,$$

$$v(s) = \lambda \int_c^d M(s,t)\, v(t)\, dt - \mu \int_c^d N(s,t)\, v(t)\, dt,$$

where the kernels are subject to certain conditions (§3), and the expansion of arbitrary functions of two variables in terms of the characteristic functions.

* Presented to the Society, October 25, 1919.

† *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen,* pp. 262–267.

‡ *Theorems of oscillation for two linear differential equations of the second order with two parameters,* these Transactions, vol. 13 (1912), pp. 22–34.

§ *Ein zweiparametriges Oscillationstheorem,* Göttinger Nachrichten, 1910, pp. 586–594.

I. Linear equations in infinitely many unknowns

1. **Existence of solutions.** Consider the system of linear equations (1) where the matrices K, L, M, N are real and symmetric, the sum of the squares of the elements in each matrix is convergent, the two matrices L and N are positive definite and $k_{ij}n_{kl}+l_{ij}m_{kl}\not\equiv 0$. We shall show that under these conditions there exist values of λ and μ for which the equations (1) have solutions of finite norm and not identically zero, that is, neither $\{u_i\}$ nor $\{v_k\}$ is identically zero. The method consists of the elimination of μ from (1), the transformation of the resulting system of equations in the one parameter λ into a system of the form (9) for which the existence of characteristic numbers λ_i has already been established, the substitution of these λ_i in one system of equations of (1), and the proof of the existence of corresponding characteristic numbers $\mu_{i\alpha}$, and finally the proof that λ_i and $\mu_{i\alpha}$ are also characteristic numbers for the other equations of (1).

The elimination of μ in (1) gives the system

$$u_i\sum_l n_{kl}v_l + v_k\sum_j l_{ij}u_j = \lambda\Big(\sum_j k_{ij}u_j\sum_l n_{kl}v_l + \sum_j l_{ij}u_j\sum_l m_{kl}v_l\Big). \tag{2}$$

Since L and N are positive definite their characteristic numbers are positive; call them α_i^2 and β_k^2. The matrices formed from the corresponding solutions $\overset{*}{l}_{ij}$ and $\overset{*}{n}_{kl}$ are orthogonal,* and hence the system (2) is equivalent to

$$\overset{*}{u}_i\overset{*}{v}_k\left(\frac{1}{\alpha_i^2}+\frac{1}{\beta_k^2}\right)=\lambda\sum_{j,l}\left(\overset{*}{k}_{ij}\frac{\delta_{kl}}{\beta_k^2}+\overset{*}{m}_{kl}\frac{\delta_{ij}}{\alpha_i^2}\right)\overset{*}{u}_j\,\overset{*}{v}_l, \tag{3}$$

where

$$\begin{aligned} &\overset{*}{u}_i=\sum_j \overset{*}{l}_{ij}u_j \quad , \quad \overset{*}{v}_k=\sum_l \overset{*}{n}_{kl}v_l,\\ &K^*=L^*KL^{*\prime} \quad , \quad M^*=N^*MN^{*\prime},\\ &\delta_{ij}=1 \text{ if } i=j, \text{ and } =0, \text{ if } i\neq j. \end{aligned} \tag{4}$$

With the substitution

$$\begin{aligned} &\overset{*}{c}_\xi=\overset{*}{u}_i\overset{*}{v}_k,\\ &a_{\xi\eta}=\overset{*}{k}_{ij}\frac{\delta_{kl}}{\beta_k^2}+\overset{*}{m}_{kl}\frac{\delta_{ij}}{\alpha_i^2},\\ &c_\xi^2=\frac{1}{\alpha_i^2}+\frac{1}{\beta_k^2}, \end{aligned} \tag{5}$$

* For notation and terminology see Hilbert, loc. cit., Kap. XI, and Hellinger, E., *Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen*, Journal für Mathematik, vol. 136, pp. 210–262.

the system (3) becomes

$$c_\xi^2 x_\xi^* = \lambda \sum_\eta a_{\xi\eta} x_\eta^*. \tag{6}$$

The matrix A is symmetric, for the interchange of i with j and of k with l, or the interchange of ξ with η, leaves A unchanged. The matrix

$$\frac{a_{\xi\eta}}{c_\xi^2} = k_{ij}^* \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2}\delta_{kl} + m_{kl}^* \frac{\beta_k^2}{\alpha_i^2 + \beta_k^2}\delta_{ij}$$

is completely continuous. Let $x_\xi = x_{ik}$ be of norm $\leqq 1$. Since

$$\frac{\alpha_i^2}{\alpha_i^2 + \beta_l^2} < 1,$$

$$\left| \sum_{i,j,k} x_{ik} k_{ij}^* \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} x_{jk} \right| \leqq \sqrt{\sum_{i,j} k_{ij}^{*2} \sum_{i,k} x_{ik}^2}$$

and

$$\left| \sum_{i,j,k=1}^{\infty} x_{ik} k_{ij}^* \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} x_{jk} - \sum_{i,j,k=1}^{n} x_{ik} k_{ij}^* \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} x_{jk} \right| \leqq \sqrt{\sum_{i,j=n+1}^{\infty} k_{ij}^{*2}}.$$

By hypothesis $\Sigma_{ij} k_{ij}^{*2}$ converges, and $\lim n \rightarrow 8\ \Sigma_{i,j=n}^{\infty} k_{ij}^{*2} = 0$. Hence the matrix $k_{ij}^* \delta_{kl} \alpha_i^2/(\alpha_i^2 + \beta_k^2)$ is completely continuous, and similarly for $m_{kl}^* \delta_{ij} \beta_k^2/(\alpha_i^2 + \beta_k^2)$. This shows that $a_{\xi\eta}/c_\xi^2$ is completely continuous. From the inequality

$$\left| \frac{a_{\xi\eta}}{c_\xi c_\eta} \right| \leqq \left| a_{\xi\eta} \right| \left(\frac{1}{c_\xi^2} + \frac{1}{c_\eta^2} \right)$$

it follows that $a_{\xi\eta}/c_\xi c_\eta$ is completely continuous. The matrix $a_{\xi\eta}/c_\xi c_\eta$ is symmetric, and not identically zero since $KN + LM \not\equiv 0$, hence there exist* values of λ for which the system

$$y_\xi = \lambda \sum_\eta \frac{a_{\xi\eta}}{c_\xi c_\eta} y_\eta \tag{7}$$

has solutions of finite norm. From this it follows that the system

$$z_\xi = \lambda \sum_\eta \frac{a_{\xi\eta}}{c_\eta^2} z_\eta \tag{8}$$

* Hilbert, loc. cit., p. 148.

has solutions $\{z_\xi = y_\xi c_\xi\}$ of finite norm for the same value of λ. Since the matrix $a_{\xi\eta}/c_\xi^2$ is completely continuous the adjoint* system

$$x_\xi^* = \lambda \sum_\eta \frac{a_{\xi\eta}}{c_\xi^2} x_\eta^* \tag{9}$$

also has solutions $\{x_\xi^*\}$ of finite norm. For each value of λ there is only a finite number of linearly independent solutions of (8) and an equal number of linearly independent solutions of (9), and the relation between the solutions of (8) and (9) is

$$z_\xi = c_\xi^2 x_\xi^*. \tag{10}$$

A solution $\{x_\xi^*\}$ of (9) leads to a solution $\{x_{ik}\}$, such that $\Sigma_{i,k} x_{ik}^2$ is convergent, of the system

$$\sum_i l_{ij} x_{jk} + \sum_l n_{kl} x_{il} = \lambda \sum_{j,l} (k_{ij} x_{jl} n_{kl} + l_{ij} x_{jl} m_{kl}). \tag{11}$$

We shall show later that x_{ik} either has the form $u_i v_k$, or is a sum of such terms.

Return to the system (1), and let λ be a value for which the system (11) has a solution x_{ik}. Assume first that u_i^* is the only solution of

$$u_i^* = \lambda \sum_j k_{ij} u_j^* \tag{12}$$

with the understanding that $\{u_i^*\}$ may be $= 0$, and that $\Sigma_i u_i^{*2} = 1$ if u_i^* is not $= 0$. There exists a limited matrix† P such that

$$P = \lambda K + \lambda PK - U^* U^{*\prime},$$

where

$$u_{ik}^* = u_i^* \text{ if } k = 1,\ u_{ik}^* = 0 \text{ if } k \neq 1.$$

If the system

$$U(x) = \lambda K U(x) + \mu L U(x) \tag{13}$$

has a solution $U(x)$ besides $U^*(x)$ for $\mu = 0$, then $U(x)$ satisfies

$$\begin{aligned} U(x) &= \mu\{LU(x) + PLU(x)\} + cU^*(x), \\ (U^*, LU) &= 0. \end{aligned} \tag{14}$$

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* Hilbert, loc. cit., p. 165.

† Hurwitz, W. A., *On the pseudo-resolvent to the kernel of an integral equation*, these Transactions, vol. 13 (1912), pp. 405–418.

Also if (14) has a solution $U(x)$, this solution satisfies (13), and the two equations (13) and (14) have the same solutions except for $U^*(x)$. The two equations of (14) may be expressed by one equation, for from the second part it follows that

$$c(U^*, LU^*) = -\mu[(U^*, LLU) + (U^*, LPLU)],$$

and the substitution of this value for c gives

$$(15) \qquad U(x) = \mu\left[LU(x) + PLU(x) - \frac{U^*U^{*\prime}LLU(x) + U^*U^{*\prime}LPLU(x)}{(U^*, LU^*)}\right] = \mu QU(x).$$

The equations (14) and (15) have the same solutions except for $U^*(x)$. To establish the existence of characteristic numbers of (15), consider the system

$$(16) \qquad U(x) = \mu\left\{QU(x) - \frac{LU^*U^{*\prime}LU(x) + PLU^*U^{*\prime}LU(x)}{(U^*, LU^*)}\right\} = \mu RU(x).$$

All the solutions of (15) satisfy (16), and any solution $U(x)$ of (16) which is such that

$$(17) \qquad (U^*, LU) = 0$$

satisfies (15). This condition is satisfied by all the solutions of (16) except possibly those corresponding to the value μ_0 of μ given by

$$\mu_0[(U^*, LLU^*) + (U^*, LPLU^*)] + (U^*, LU^*) = 0.$$

A solution of (16) corresponding to μ_0 is $U^*(x)$, the solution of (13) for $\mu = 0$. By adding multiples of $U^*(x)$ any other solution of (16) corresponding to μ_0 can be made to satisfy (17), and hence we may assume that the systems (13) and (16) have the same solutions. The matrix R in (16) is such that the product of R by the symmetric positive definite matrix L is symmetric, and therefore* characteristic numbers μ_α exist and are real, $\{LU_\alpha(x)\}$ is the adjoint system of the system of solutions $U_\alpha(x)$ and the matrix R may be expressed in terms of the solutions in the following form:

$$(18) \qquad R'LF(x) = \sum_\alpha \frac{LU_\alpha(x)(LF, U_\alpha)}{\mu_\alpha},$$

* A. J. Pell, *Linear equations with unsymmetric systems of coefficients*, these Transactions, vol. 20 (1919), pp. 23–39.

where $F(x)$ is any limited linear form. From this expansion it follows not only that (16) and (13) have solutions besides $U^*(x)$, but also that the system of all the solutions $U_\alpha(x)$ of the equations

$$U_\alpha(x)=\lambda KU_\alpha(x)+\mu_\alpha LU_\alpha(x) \tag{19}$$

is such that the system $\{LU_\alpha(x)\}$ is complete. Suppose that $F(x)$ is such that

$$(F, LU_\alpha)=0;$$

then from (16) and the expansion for R the linear form

$$G(x)=LF(x)+PLF(x)$$

is such that

$$LG(x)-\frac{LU^*(x)(G,LU^*)}{(U^*,LU^*)}=0.$$

The only $G(x)$ which satisfies this is $G(x) = kU^*(x)$. From the definition of the matrix P it follows that if

$$LF(x)+PLF(x)=kU^*(x),$$

then $LF(x) = cU^*(x)$, and therefore $F(x) = 0$. Hence the system $\{LU_\alpha(x)\}$ is complete.

If the system (12) had more than one linearly independent solution, similar considerations would show that again the system $\{LU_\alpha(x)\}$ is a complete system.

From (11) we obtain solutions $V_\alpha(x)$ of the second system of equations of (1) by multiplying by U_α,

$$V_\alpha(x)=\lambda MV_\alpha(x)-\mu_\alpha NV_\alpha(x), \tag{20}$$

for the characteristic numbers λ and μ_α. The solutions $V_\alpha(x)$ are given by

$$V_\alpha(x)=X'LU_\alpha(x),$$

and since $\{LU_\alpha(x)\}$ is complete the $V_\alpha(x)$ are not all zero, and the equations (19) and (20) have solutions not identically zero. We have established the following theorem.

THEOREM 1. *If the matrices K, L, M, and N are symmetric matrices such that the sum of the squares of the elements in each is convergent, if L and N are positive definite, and if $k_{ij}n_{kl}+l_{ij}m_{kl}\not\equiv 0$, there exist real values of λ and μ for which the system* (1) *has solutions u_i, v_k of finite norm and not identically zero.*

2. **Expansion of the determinant matrix.** Since every solution $u_{\alpha i}$, $v_{\alpha k}$ of finite norm of (19) and (20) satisfies (9), only a finite number of $v_{\alpha k}$ can be different from zero. It has been shown that $\{LU_\alpha(x)\}$ is a complete system, and therefore

$$x_{ik}=\sum_{\alpha=1}^{n} u_{\alpha i}v_{\alpha k},$$

for otherwise the difference between the left and right hand sides would be orthogonal to $LU_\alpha(x)$. We may therefore assume that the solutions of (11) are in the form of products $u_{\alpha i}v_{\alpha k}$.

From the expansion of the symmetric completely continuous matrix* $(a_{\xi\eta}/c_\xi c_\eta)$,

$$\frac{a_{\xi\eta}}{c_\xi c_\eta} = \sum_\alpha \frac{y_{\alpha\xi}y_{\alpha\eta}}{\lambda_\alpha},$$

it follows that

$$a_{\xi\eta} = \sum_\alpha \frac{c_\xi^2 x_{\alpha\xi}^* c_\eta^2 x_{\alpha\eta}^*}{\lambda_\alpha},$$

and this gives the following expansion for the determinant matrix:

$$k_{ij}n_{kl} + l_{ij}m_{kl} = \sum_\alpha \frac{w_{\alpha ik}w_{\alpha jl}}{\lambda_\alpha}, \tag{21}$$

where

$$w_{\alpha ik}=v_{\alpha k}\sum_j l_{ij}u_{\alpha j} + u_{\alpha i}\sum_l n_{kl}v_{\alpha l}.$$

The following theorem has been established.

THEOREM 2. *Under the conditions of Theorem* 1, *the determinant matrix* $KN + LM$ *may be expressed in terms of the solutions of* (1) *in the form* (21).

II. Two linear integral equations with two parameters

3. **Existence of solutions of two integral equations.** In the two linear integral equations

$$\begin{aligned} u(x) &=\lambda \int_a^b K(x,y)u(y)dy + \mu \int_a^b L(x,y)u(y)dy, \\ v(s) &= \lambda \int_c^d M(s,t)v(t)dt - \mu \int_c^d N(s,t)v(t)dt, \end{aligned} \tag{22}$$

* Hilbert, loc. cit., p. 148.

let the kernels K, L be real symmetric functions continuous in the real variables x and y for $a \leqq x \leqq b$ and $a \leqq y \leqq b$, M, N real, symmetric and continuous in the real variables s and t for $c \leqq s \leqq d$, $c \leqq t \leqq d$, $KN + LM \not\equiv 0$, and L and N positive definite in the sense that there exist no functions $f(x), g(s)$ not identically zero which together with their squares are integrable in the sense of Lebesgue on (a, b) and (c, d) respectively, and such that $\int\int f(x)L(x,y)f(y) dxdy \leqq 0$, $\int\int g(s)N(s,t)g(t)dsdt \leqq 0$. Let $\{\varphi_i(x)\}$ be a closed orthogonal system of continuous functions for the interval (a, b), and $\{\psi_k(s)\}$ for the interval (c, d). Multiply the equations (22) by $\varphi_i(x)$ and $\psi_k(s)$, respectively, integrate, and then expand the right hand sides;

$$(23)\quad \begin{aligned} \int \varphi_i u &= \lambda \sum_j \int\int \varphi_i K \varphi_j \int \varphi_j u + \mu \sum_j \int\int \varphi_i L \varphi_j \int \varphi_j u, \\ \int \psi_k v &= \lambda \sum_l \int\int \psi_k M \psi_l \int \psi_l v - \mu \sum_l \int\int \psi_k N \psi_l \int \psi_l v. \end{aligned}$$

If the equations (22) have continuous solutions $u(x)$, $v(s)$, then the system (23) has solutions of finite norm. The matrices in (23) satisfy all the conditions imposed on the matrices in Theorem 1, and therefore there exist values of λ and μ, necessarily real, for which the system (23) has solutions x_i, y_k of finite norm. In the usual way it can be shown that the functions

$$u(x) = \lambda \sum_i x_i \int K\varphi_i + \mu \sum_i x_i \int L\varphi_i,$$

$$v(s) = \lambda \sum_k y_k \int M\varphi_k - \mu \sum_k y_k \int N\psi_k$$

are continuous in (a, b) and (c, d), respectively, and satisfy the equations (22).

THEOREM 3. *If the kernels K, L, M, N are continuous real symmetric functions, L and N positive definite, and $KN + LM \not\equiv 0$, there exist values of λ and μ necessarily real and for which the equations* (22) *have continuous solutions $u(x)$, $v(s)$, not identically zero.*

4. **Properties of the solutions.** Let $u_i(x)$, $v_i(s)$ be solutions of (22) corresponding to the parameter values λ_i, μ_i, and $u_k(x)$, $v_k(s)$ to λ_k, μ_k. Since the kernels are symmetric it follows from (22) that

$$
(24)\qquad
\begin{aligned}
\int u_i u_k &= \lambda_i \int\int u_i K u_k + \mu_i \int\int u_i L u_k \\
&= \lambda_k \int\int u_i K u_k + \mu_k \int\int u_i L u_k, \\
\int v_i v_k &= \lambda_i \int\int v_i M v_k - \mu_i \int\int v_i N v_k \\
&= \lambda_k \int\int v_i M v_k - \mu_k \int\int v_i N v_k.
\end{aligned}
$$

By subtraction we obtain

$$
(25)\qquad
\begin{aligned}
(\lambda_i - \lambda_k) \int\int u_i K u_k + (\mu_i - \mu_k) \int\int u_i L u_k = 0, \\
(\lambda_i - \lambda_k) \int\int v_i M v_k - (\mu_i - \mu_k) \int\int v_i N v_k = 0.
\end{aligned}
$$

If the two parameter sets are not identical the determinant of the coefficients must equal zero:

$$
(26)\qquad
\begin{vmatrix}
\int\int u_i K u_k & \int\int u_i L u_k \\
\int\int v_i M v_k & -\int\int v_i N v_k
\end{vmatrix} = 0.
$$

This relation together with (24) gives the result that if (u_i, v_i), (u_k, v_k) belong to different parameter sets, the matrix

$$
(27)\qquad
\begin{pmatrix}
\int u_i u_k & \int\int u_i K u_k & \int\int u_i L u_k \\
\int v_i v_k & \int\int v_i M v_k & -\int\int v_i N v_k
\end{pmatrix}
$$

is of rank less than 2. If the solutions belong to the same values of the parameters, linear combinations may be formed so that this condition is satisfied.

Since the kernels L and N are positive definite the solutions u_i, v_i may be multiplied by constants such that

$$
\int u_i^2 \int\int v_i N v_i + \int v_i^2 \int\int u_i L u_i = 1.
$$

In the following we shall assume that the solutions have been so normalized.

Consider any set of functions $u_i(x)$, $v_i(s)$ which are continuous on (a, b), (c, d), respectively, and have the property that

$$\int u_i u_k \int\int v_i N v_k + \int v_i v_k \int\int u_i L u_k = \delta_{ik}. \tag{28}$$

If the series

$$f(x, s) = \sum_i A_i u_i(x) v_i(s) \tag{29}$$

is uniformly convergent the coefficients A_i may be determined in terms of $f(x, s)$. Multiply the equality (29) by

$$v_k(s) \int L(x, y) u_k(y) dy + u_k(x) \int N(s, t) v_k(t) dt$$

and integrate; then on account of (28).

$$A_k = \int\int\int u_k(x) L(x,y) f(y,s) v_k(s) dx dy ds$$
$$+ \int\int\int u_k(x) f(x,s) N(s,t) u_k(t) dx ds\, dt.$$

Similarly if the series

$$f(x, s) = \sum_i B_i [v_i(s) \int u_i(y) L(y,x) dy + u_i(x) \int N(s,t) v_i(t) dt]$$

is uniformly convergent,

$$B_i = \int\int u_i(x) f(x,s) v_i(s) dx ds.$$

Let $f(x, s)$ be any function continuous in x on (a, b) and in s on (c, d), and let

$$f_i = \int\int\int u_i(x) L(x,y) f(y,s) v_i(s) dx dy ds + \int\int\int u_i(x) f(x,s) N(s,t) v_i(t) dx ds dt.$$

Call f_i the Fourier coefficients of $f(x, s)$ with respect to u_i, v_i. If (29) is uniformly convergent,

$$\sum_i f_i g_i = \int\int\int f(x,s) L(x,y) g(y,s) dx dy ds + \int\int\int g(x,s) N(s,t) f(x,t) dx ds dt,$$

where $g(x, s)$ is a continuous function.

The Fourier coefficients of any continuous function $f(x, s)$ with respect to a set of functions u_i, v_i with the property (28) are of finite norm. For

$$(30)\quad \int\int\int F(x,s)L(x,y)F(y,s)dxdyds + \int\int\int F(x,s)N(s,t)F(x,t)dxdsdt \geqq 0,$$

and if

$$F(x,s) = f(x,s) - \sum_{i=1}^{n} f_i u_i(x) v_i(s),$$

the inequality (30) reduces to

$$\int\int\int f(x,s)L(x,y)f(y,s)dxdyds + \int\int\int f(x,s)N(s,t)f(x,t)dxdsdt$$

$$- 2\sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n} f_i^2 \geqq 0.$$

Therefore

$$(31)\quad \sum_i f_i^2 \leqq \int\int\int f(x,s)L(x,y)f(y,s)dxdyds + \int\int\int f(x,s)N(s,t)f(x,t)dxdsdt,$$

which shows that the sequence $\{f_i\}$ is of finite norm.

5. **Expansion of arbitrary functions of two variables.** A consequence of (23) and (21) is

$$(32)\quad \int\int\int\int [K(x,y)f(x,s)N(s,t)g(y,t) + L(x,y)f(x,s)M(s,t)g(y,t)]dxdydsdt$$

$$= \sum_\alpha \frac{f_\alpha g_\alpha}{\lambda_\alpha},$$

where $f(x, s)$ and $g(x, s)$ are any two continuous functions, and therefore if the series on the right is uniformly convergent,

$$K(x,y)N(s,t) + L(x,y)M(s,t) = \sum_\alpha \frac{W_\alpha(x,s)W_\alpha(y,t)}{\lambda_\alpha},$$

where

$$W_\alpha(x,s) = v_\alpha(s)\int L(x,y)u_\alpha(y)dy + u_\alpha(x)\int N(s,t)v_\alpha(t)dt.$$

To obtain the expansion of arbitrary functions, we first show that

$$\left\{\frac{W_\alpha(x,s)}{\lambda_\alpha}\right\}$$

is of finite norm; this would follow immediately if there were a continuous function $F(x, y, s, t)$ such that

$$\int\int\int\int F(x,y,s,t)\varphi_i(x)\varphi_j(y)\psi_k(s)\psi_l(t)dxdydsdt = b_{ikjl} = a_{ikjl}\frac{\alpha_j^2\beta_l^2}{\alpha_j^2+\beta_l^2},$$

where

$$a_{ijkl} = \int\int K(x,y)\varphi_i(x)\varphi_j(y)dxdy\,\frac{\delta_{kl}}{\beta_k^2} + \int\int M(s,t)\psi_k(s)\psi_l(t)dsdt\,\frac{\delta_{ij}}{\alpha_i^2},$$

and $\varphi_i(x)$ are the characteristic functions of $L(x,y)$ corresponding to the characteristic numbers α_i^2, and $\psi_k(s)$ the characteristic functions of $N(s, t)$ corresponding to the characteristic numbers β_k^2, for then

$$W_\alpha(x,s) = \lambda_\alpha\int\int F(x,y,s,t)W_\alpha(y,t)dydt,$$

and (31) would give the convergence of

$$\sum_\alpha \frac{W_\alpha^2(x,s)}{\lambda_\alpha^2}.$$

Denote by $F[f(x, s)]$ the transformation defined by

$$\int\int F[f(x,s)]\varphi_i(x)\psi_k(s)dxds = \sum_{j,l} b_{ikjl}\int\int f(x,s)\varphi_j(x)\psi_l(s)dxds.$$

If the function $F(x, y, s, t)$ above exists, then

$$F[f(x,s)] = \int\int F(x,y,s,t)f(y,t)dydt.$$

The transformed function of $g(s)\int L(x, y)\, f(y)dy$ exists and is a continuous function, for it may be expressed by

$$F[g(s)\int L(x,y)f(y)dy] = \sum_{i,k}\frac{\int K(x,y)\varphi_i(y)dy\int f\varphi_i}{\alpha_i^2}\,\frac{\alpha_i^2}{\alpha_i^2+\beta_k^2}\psi_k(s)\int g\psi_k$$
$$+\sum_{i,k}\frac{\varphi_i(x)\int f\varphi_i}{\alpha_i^2}\left(\frac{\beta_k^2}{\alpha_i^2+\beta_k^2}\int g\psi_k\int M(s,t)\psi_k(t)dt\right),$$

a series with continuous terms and such that the series formed from the absolute values of the terms is uniformly convergent.* Similarly the transformed func-

* Mercer, J., *Functions of positive and negative type,* Transactions of the Royal Society, A, vol. 209 (1909), pp. 415–446.

tion of $f(x)\int N(s, t)\, g(t)\, dt$ exists and is continuous. From the previous work it follows that the transformed function of $W_\alpha(x, s)$ is given by

$$F[W_\alpha(x,s)] = \frac{W_\alpha(x,s)}{\lambda_\alpha}. \tag{33}$$

From the series which represents $F[g(s)\int Lf]$ the following inequality is obtained:

$$2|F[g\int Lf]| \leqq \sum_{i,k} \frac{\left(\int f\varphi_i\right)^2}{\alpha_i^2}\left(\int g\psi_k\right)^2 + \Phi^2(x,s),$$

and hence

$$2|F[g\int Lf]| \leqq \int fLf\int g^2 + \Phi^2(x,s), \tag{34}$$

where

$$\Phi^2(x,s) = 2\sum_{i,k}\left(\int K(x,y)\varphi_i(y)dy\right)^2 \frac{\psi_k^2(s)}{\alpha_i^2+\beta_k^2} + \frac{\varphi_i^2(x)}{\alpha_i^2}\left(\int M(s,t)\psi_k(t)dt\right)^2$$

and is a continuous function in x and s. Similarly it may be shown that

$$2|F[f\int Ng]| \leqq \int gNg\int f^2 + \Psi^2(x,s), \tag{35}$$

where $\Psi^2(x, s)$ is a continuous function in x and s. From the two inequalities (34) and (35) it follows that

$$\begin{aligned}\Phi^2(x,s) - 2F[\int L(x,y)f(y,s)dy] + \int\int\int f(x,s)L(x,y)f(y,s)dxdyds \\ + \Psi^2(x,s) - 2F[\int N(s,t)f(x,t)dt] + \int\int\int f(y,s)N(s,t)f(y,t)dydsdt \geqq 0\end{aligned} \tag{36}$$

where

$$f(x,s) = \sum_{j=1}^{n} F_j u_j(x)v_j(s)$$

and

$$F_j = \frac{W_j(x,s)}{\lambda_j},$$

or by virtue of (33)

$$F_j = F[v_j(s)\int L(x,y)u_j(y)dy + u_j(x)\int N(s,t)v_j(t)dt].$$

The inequality (36) reduces to

$$\Phi^2(x,s) - 2\sum_{j=1}^{n} F_j^2 + \sum_{j=1}^{n} F_j^2 + \Psi^2(x,s) \geqq 0,$$

and therefore

$$\sum_{j=1}^{n} F_j^2 = \sum_{j=1}^{n} \frac{W_j^2(x,s)}{\lambda_j^2} \leqq \Phi^2(x,s) + \Psi^2(x,s).$$

This last result and the inequality (31) give the uniform convergence of the series formed from the absolute values of the terms of the series

$$\sum_{\alpha} \frac{W_\alpha(x,s) f_\alpha}{\lambda_\alpha},$$

where $f(x, s)$ is any continuous function, and from (32) it follows that

$$h(x,s) = \int\int (K(x,y)f(y,t)N(s,t) + L(x,y)f(y,t)M(s,t))dydt$$
$$= \sum_{\alpha} \frac{W_\alpha(x,s) f_\alpha}{\lambda_\alpha} = \sum_{\alpha} W_\alpha(x,s) \int h u_\alpha v_\alpha.$$

THEOREM 4. *If the kernels K, L, M, N are continuous real symmetric functions, L and N positive definite in the sense defined in* §3, *and* $K\,N + L\,M \not\equiv 0$, *any function $h(x, s)$ which can be expressed in the form*

$$h(x,s) = \int\int (K(x,y)f(y,t)N(s,t) + L(x,y)f(y,t)M(s,t))dydt,$$

where $f(x, s)$ is any continuous function, may be expanded into the uniformly and absolutely convergent series

$$h(x,s) = \sum_{\alpha} W_\alpha(x,s) \int h u_\alpha v_\alpha.$$

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